Partially coherent fields, the transport-of-intensity equation, and phase uniqueness

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Recent papers have shown that there are different coherent and partially coherent fields that may have identical intensity distributions throughout space. On the other hand, the well-known transport-of-intensity equation allows the phase of a coherent field to be recovered from intensity measurements, and the solution is widely held to be unique. A discussion is given on the recovery of the structure of both coherent and partially coherent fields from intensity measurements, and we reconcile the uniqueness question by showing that the transport-of-intensity equation has a unique solution for the phase only if the intensity distribution has no zeros.

1. INTRODUCTION

In the past ten years there has been considerable interest in the determination of phase or coherence properties of light from measurements of its intensity distribution.[1–4] The coherent phase recovery is of most direct interest in the field of adaptive optics,[5,6] whereas the recovery of the coherence properties of partially coherent radiation[7–9] is of importance in microscopy and atom optics, where it is often inevitable or desirable that fields are partially coherent.

Of particular importance is the work of Teague,[1,2] who showed that two displaced intensity measurements should allow the phase of a coherent wave to be determined with the so-called transport-of-intensity equation. Recently, Gori *et al.*[10] have presented an example demonstrating that it is possible for distinct coherent wave fields to have identical three-dimensional intensity distributions. It follows that linear combinations of these waves will have identical three-dimensional intensity distributions but different coherence properties. The example found by Gori *et al.* raises important questions concerning the uniqueness of the solutions of the transport-of-intensity equation. In particular, Teague claimed to have proved that the transport-of-intensity equation has a unique solution for the phase distribution. A major aim of the present paper is to reconcile these two results.

In Section 2 of this paper we use the well-known transport equations for generalized radiance[11] to derive the transport of intensity for light with arbitrary coherence and use this to show that the solutions to this equation will not be unique. We then specialize the result to obtain the coherent transport-of-intensity equation first presented by Teague. We believe that the physical picture corresponding to this formalism allows some insight into the nature of the phase problem, and we exploit this picture throughout this paper.

In Section 3 we discuss issues concerning the uniqueness of the solutions to the transport-of-intensity equation in order to examine the apparent contradiction between the work of Gori *et al.* and the work of Teague. We show that the resolution lies in the fact that the examples of Gori *et al.* contain points of zero intensity, where the phase is not defined and the intensity transport equation is not valid. The presence of zero-intensity points leads to the branching of the phase and the appearance of different phase solutions corresponding to an identical intensity distribution.

2. GENERALIZED RADIANCE AND TRANSPORT OF INTENSITY

A. Partially Coherent Transport-of-Intensity Equation

Consider partially coherent quasi-monochromatic light described by the mutual optical intensity function $J(\mathbf{r}_1, \mathbf{r}_2)$. We introduce the variables

$$\mathbf{r} \equiv 1/2(\mathbf{r}_1 + \mathbf{r}_2), \qquad \mathbf{x} \equiv (\mathbf{r}_1 - \mathbf{r}_2)\,, \tag{1}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are contained in the plane perpendicular to the optic axis. Distance along the optic axis is denoted by z. The generalized radiance (GR)[11] is defined by means of a Fourier transform over the $\mathbf{x}$ variable:

$$B(\mathbf{r}, \mathbf{u}, z) = \frac{1}{\lambda^2} \int J(\mathbf{r}, \mathbf{x}, z)\exp(-2\pi i \mathbf{x} \cdot \mathbf{u}/\lambda)\mathrm{d}\mathbf{x}\,. \tag{2}$$

If we substitute this back into the paraxial propagation expressions for the mutual optical intensity function,[12] then we find the following very simple geometric propagation expression for the GR:

$$B(\mathbf{r}, \mathbf{u}, z) = B(\mathbf{r} - z\mathbf{u}, \mathbf{u}, 0)\,. \tag{3}$$

The intensity is obtained by integration over $\mathbf{u}$:

$$I(\mathbf{r}, z) = \int B(\mathbf{r}, \mathbf{u}, z)\mathrm{d}\mathbf{u}\,. \tag{4}$$

Careful inspection of these equations leads to the conclusion that they are precisely identical to those describing geometric optics if $B(\mathbf{r}, \mathbf{u}, z)$ is interpreted as describing the distribution of energy flow as a function of position $\mathbf{r}$ and direction $\mathbf{u}$. The vector $\mathbf{u}$ is then interpreted as

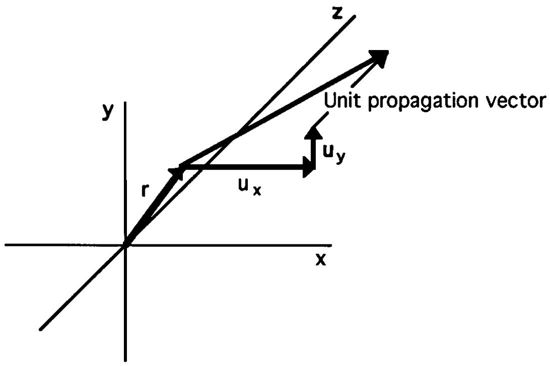

Fig. 1. Propagation geometry that suggests a simple ray interpretation of the generalized radiance. As noted in the text, this interpretation is subject to some important limitations.

the projection of the unit vector specifying the direction of propagation of the ray at the location $\mathbf{r}$ onto the plane perpendicular to the optic axis (see Fig. 1). Although this interpretation is intuitively appealing, it must be tempered by the recognition that, although $B(\mathbf{r}, \mathbf{u}, z)$ is real, it may sometimes be negative. The negativity of $B(\mathbf{r}, \mathbf{u}, z)$ prevents its unambiguous interpretation as a description of energy flow. In the short-wavelength limit, B is real and positive,[13] and the interpretation of B as energy flow then appears unambiguous and appropriate.

With these results we see that the intensity at some plane a distance z from the $z = 0$ plane is given by

$$I(\mathbf{r}, z) = \int B(\mathbf{r} - z\mathbf{u}, \mathbf{u}, 0)\mathrm{d}\mathbf{u}\,. \tag{5}$$

Thus the rate of change of intensity along z is given by

$$\frac{\partial I}{\partial z} = \frac{\partial}{\partial z}\int B(\mathbf{r} - z\mathbf{u}, \mathbf{u}, 0)\mathrm{d}\mathbf{u}\,. \tag{6}$$

If we define the plane of interest to be the $z = 0$ plane, then we find that

$$\frac{\partial I}{\partial z} = -\nabla_{\mathbf{r}} \cdot \int \mathbf{u}B(\mathbf{r}, \mathbf{u}, 0)\mathrm{d}\mathbf{u}\,, \tag{7}$$

where $\nabla_{\mathbf{r}} = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j}$ is the two-dimensional gradient operator. Equation (7) is a simple extension of transport equations for the Wigner distribution that have been published elsewhere[11] and may be regarded as the transport-of-intensity equation for a paraxial partially coherent field. Since $B(\mathbf{r}, \mathbf{u}, z)$ is the phase-space density function for paraxial optics, Eq. (7) follows directly from the conservation of phase-space density as expressed by Liouville's theorem.

It is easy to show that Eq. (7) may have nonunique solutions for $B(\mathbf{r}, \mathbf{u}, 0)$. Any field that has the symmetry $B(\mathbf{r}, \mathbf{u}, 0) = B(\mathbf{r}, -\mathbf{u}, 0)$ will result in the integral on the right-hand side of Eq. (7) vanishing, with the result that

$$\frac{\partial I}{\partial z} = 0\,. \tag{8}$$

There are many partially coherent optical fields that may have this property. Any solution to Eq. (7) may have such a symmetric function added to it and still be a valid solution. Thus we conclude that it will not be possible to reconstruct unambiguously a partially coherent field from measurements of its intensity distribution and its first derivative by means of the partially coherent transport-of-intensity equation.

B. Coherent Transport-of-Intensity Equation

Let us now turn to coherent waves. In what follows we obtain the familiar coherent transport-of-intensity equation as a special case of Eq. (7). This derivation is more complex than the derivation presented by Teague, but it does, however, allow a clear physical interpretation to be drawn about the nature of phase retrieval with this approach.

In the limit of a coherent wave the mutual optical intensity function may be written as

$$J(\mathbf{r}, \mathbf{x}, z) = \psi(\mathbf{r} + \mathbf{x}/2, z)\psi^*(\mathbf{r} - \mathbf{x}/2, z)\,, \tag{9}$$

where $\psi(\mathbf{r})$ is the complex field of the coherent wave. We wish to find the GR corresponding to this field in the paraxial limit. Let us write

$$\psi(\mathbf{r}, z) = A(\mathbf{r}, z)\exp[2\pi i\phi(\mathbf{r}, z)]\,, \tag{10}$$

where $A(\mathbf{r}, z) \equiv \sqrt{I(\mathbf{r}, z)}$. We substitute this into Eq. (9) and then into the definition of the GR to obtain

$$\begin{aligned} B_{\mathrm{coh}}(\mathbf{r}, \mathbf{u}, z) = \frac{1}{\lambda^2}\int & A(\mathbf{r} + \mathbf{x}/2, z)A(\mathbf{r} - \mathbf{x}/2, z) \\ & \times \exp\{2\pi i[\phi(\mathbf{r} + \mathbf{x}/2, z) \\ & - \phi(\mathbf{r} - \mathbf{x}/2, z)]\}\exp(-2\pi i\mathbf{x}\cdot\mathbf{u}/\lambda)\mathrm{d}\mathbf{x}\,. \end{aligned} \tag{11}$$

Let us now Taylor-expand the term in the first exponent and assume that the phase ϕ varies sufficiently slowly with $\mathbf{x}$ that terms of order $\nabla_{\mathbf{r}}{}^3\phi(\mathbf{r}, z)$ can be ignored. This term then reduces to

$$\phi(\mathbf{r} + \mathbf{x}/2, z) - \phi(\mathbf{r} - \mathbf{x}/2, z) \approx \mathbf{x}\cdot\nabla_{\mathbf{r}}\phi(\mathbf{r}, z)\,. \tag{12}$$

Note that the assumption leading to this result is also implicit in the paraxial approximation that we adopted above.

If we define

$$\begin{aligned} B_a(\mathbf{r}, \mathbf{u}, 0) \equiv \frac{1}{\lambda^2}\int & A(\mathbf{r} + \mathbf{x}/2, 0)A(\mathbf{r} - \mathbf{x}/2, 0) \\ & \times \exp(-2\pi i\mathbf{x}\cdot\mathbf{u}/\lambda)\mathrm{d}\mathbf{x}\,, \end{aligned} \tag{13}$$

then $B_a(\mathbf{r}, \mathbf{u}, 0)$ is the GR corresponding only to the amplitude distribution across the plane at $z = 0$. We now apply the convolution theorem to Eq. (11) to find that the GR corresponding to a general coherent wave in the paraxial approximation is described by

$$B(\mathbf{r}, \mathbf{u}, 0) = B_a[\mathbf{r}, \mathbf{u}' - \lambda\nabla_{\mathbf{r}}\phi(\mathbf{r}, 0)]\,, \tag{14}$$

so that

$$\frac{\partial I}{\partial z} = -\nabla_{\mathbf{r}} \cdot \int \mathbf{u}B_a[\mathbf{r}, \mathbf{u} - \lambda\nabla_{\mathbf{r}}\phi(\mathbf{r}, 0)]\mathrm{d}\mathbf{u}\,, \tag{15}$$

which may be written as

$$\frac{\partial I}{\partial z} = -\nabla_{\mathbf{r}} \cdot \int [\mathbf{u}' + \lambda\nabla_{\mathbf{r}}\phi(\mathbf{r}, 0)]B_a(\mathbf{r}, \mathbf{u}', 0)\mathrm{d}\mathbf{u}'\,. \tag{16}$$

It is easily shown that $I(\mathbf{r}, 0) = \int B_a(\mathbf{r}, \mathbf{u}', 0)\mathrm{d}\mathbf{u}'$, so, if we separate out the term involving $\lambda\nabla_\mathbf{r}\phi(\mathbf{r}, 0)$ and perform the integral over $\mathbf{u}'$, we find that

$$\frac{\partial I}{\partial z} = -\nabla_\mathbf{r} \cdot \left[\lambda\nabla_\mathbf{r}\phi(\mathbf{r}, 0)I(\mathbf{r}, 0) + \int \mathbf{u}'B_a(\mathbf{r}, \mathbf{u}', 0)\mathrm{d}\mathbf{u}'\right]. \tag{17}$$

Given that $B_a(\mathbf{r}, \mathbf{u}', 0)$ is expressed in Eq. (13) as a Fourier transform of a real and even quantity, we deduce that $B_a(\mathbf{r}, \mathbf{u}', 0) = B_a(\mathbf{r}, -\mathbf{u}', 0)$. This implies that the second term in Eq. (17) vanishes, so that we obtain

$$\frac{\partial I}{\partial z} = -\nabla_\mathbf{r} \cdot [\lambda\nabla_\mathbf{r}\phi(\mathbf{r}, 0)I(\mathbf{r}, 0)], \tag{18}$$

which is Teague's transport-of-intensity expression. This expression is analogous to differential expressions for the continuity equations in fluid dynamics and electrodynamics. The fluid analogy is emphasized if we note that $\lambda\nabla_\mathbf{r}\phi(\mathbf{r}, 0)$ is the phase velocity vector for the wave field.

From this vantage point we see that the symmetry that prevents the unambiguous determination of partially coherent fields results in a simplification of the transport-of-intensity equation. However, the remaining phase term cannot show this symmetry unless $\phi = \text{constant}$, which is, of course, also a unique solution. Thus, in the coherent case, there remains the prospect of a unique solution.

3. UNIQUENESS OF THE RECONSTRUCTED PHASE OF COHERENT FIELDS

In this section we examine the question of uniqueness of the phase reconstructed from the transport-of-intensity equation. We show that if the intensity is everywhere nonzero, then the phase will be uniquely defined, to within an additive constant. The proofs of uniqueness, however, are not valid in the case in which the wave field contains points of zero intensity, and the example of Ref. 10 presents a case in which the many solutions for the phase can exist for a given three-dimensional intensity distribution.

Let us begin by recalling the definition of phase. From Eq. (10) we can write

$$\phi(\mathbf{r}, z) = \frac{1}{2\pi}\arg[\psi(x, y)] = \frac{1}{2\pi i}\,\ell\mathrm{n}\left[\frac{\psi(\mathbf{r}, z)}{A(\mathbf{r}, z)}\right]. \tag{19}$$

Note that the phase is well defined in the case in which the amplitude of the wave is strictly positive and that, as has been described elsewhere, points of zero intensity are the branching points of the phase.[14] In Subsection 3.A we review the proofs of the uniqueness of the solutions for the phase to the relevant Dirichlet and Neuman problems and show that these proofs are valid only in the case in which the intensity is everywhere strictly positive.

In Subsection 3.B we examine the case in which the wave field is described by an analytic function, and we find the nonunique solutions of the transport-of-intensity equations for a harmonic phase function in the case in which the intensity is a circularly symmetric function. Interestingly, these turn out to be the solutions obtained by Gori *et al.* and are the only solutions to the transport-of-intensity equation that are circularly symmetric with a point of zero intensity on axis. Finally, in Subsection 3.C we present a physical picture for the nonunique phase solutions based on the GR picture of wave fields.

A. Uniqueness Proofs for the Transport-of-Intensity Equation

We now review the uniqueness proofs that exist for the transport-of-intensity equation. To do this, we first set up some mathematical preliminaries.

Let Ω be a simply connected bounded domain in two-dimensional (x, y) space with a smooth (infinitely differentiable) boundary Γ. By simply connected we mean that Ω contains no holes. Consider a partial differential operator L defined by

$$L \equiv -\nabla_\mathbf{r} \cdot (I\nabla_\mathbf{r}) \tag{20}$$

in Ω involving a smooth function $I = I(x, y)$ satisfying

$$I(x, y) > 0 \qquad \forall(x, y) \in \Omega\,. \tag{21}$$

With these definitions we may write the transport-of-intensity Eq. (18) in the form

$$L\varphi = f\,, \tag{22}$$

where $f \equiv (1/\lambda)(\partial I/\partial z)$. Given this formulation, we may use a classic result of the theory of elliptic partial differential equations,[15] which states that, for any smooth functions f in Ω and g on the boundary Γ, there exists a unique smooth solution ϕ to the Dirichlet problem for the operator L in the domain Ω:

$$L\phi = f, \qquad \phi|_\Gamma = g\,. \tag{23}$$

Hence, with a specified boundary condition, we may find a unique solution to the transport-of-intensity equation provided that the intensity distribution is strictly positive.

In some cases it is more convenient to consider what is known as the Neuman problem.[5] In this case we let the domain Ω and the functions $I(x, y)$, $f(x, y)$, and g be the same as those above. Then a smooth solution to the Neuman problem,

$$L\phi = f, \qquad I\partial_n\phi|_\Gamma = g \tag{24}$$

(here $\partial_n\phi = \mathbf{n} \cdot \nabla_\mathbf{r}\phi$ is the derivative in the direction normal to the boundary), exists if and only if the following condition holds:

$$\int_\Omega f(x, y)\mathrm{d}x\mathrm{d}y + \int_\Gamma g(s)\mathrm{d}s = 0\,. \tag{25}$$

If we substitute for f and g, Eq. (25) becomes

$$\frac{1}{\lambda}\frac{\partial}{\partial z}\int_\Omega I(x, y)\mathrm{d}x\mathrm{d}y = -\int_\Gamma I\partial_n\varphi\,\mathrm{d}s\,, \tag{26}$$

which is just an expression of conservation of energy; loss of intensity in a region arises through energy flow across the boundary of the region. If condition (25) is satisfied, the solution ϕ here is unique up to an arbitrary constant.

As foreshadowed, these uniqueness theorems rely on the intensity being strictly positive and the function $\phi(x, y)$ being a smooth single-valued function over the region Ω. A function, ϕ, representing phase, however, need not be smooth and single valued, and the wave field can contain points of zero intensity. An example of this is presented in Subsection 3.B.

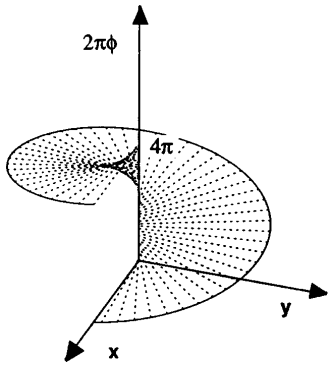

Fig. 2. Phase function $2\pi\phi = m\theta$ plotted as a function of x and y for $m = 2$.

B. Class of Nonunique Solutions

Consider the examples of Ref. 10, which have identical three-dimensional intensity distributions but different phases. Hence no method exists that would allow unique phase reconstruction from any kind of intensity measurement in free space. The wave fields $\psi_m(r, \theta, z)$ are the results of the Fresnel propagation of the initial distributions:

$$\psi_m(r, \theta, 0) = I_0^{1/2}(r)\exp(i2\pi\phi_n), \tag{27}$$

where

$$2\pi\varphi_m = m\theta \tag{28}$$

and m is an integer. This phase function is plotted as a function of θ for $m = +2$ in Fig. 2 and exhibits a helical form. Equation (28) is the well-known single vortex solution of the paraxial wave equation.

As described in Ref. 10, $\psi_m(r = 0, \theta, z) = 0$ for all values of θ and z and, according to Eq. (19), there is a phase singularity at $r = 0$. Singularities of the type described above are the subject of a great deal of interest, and the phase singularity at $r = 0$ is said to have a topological charge of m.[16,17]

Note that the phases given in Eq. (28) are harmonic functions; i.e., they satisfy $\Delta\phi = \nabla_r{}^2\phi = 0$. The class of wave fields possessing harmonic phase functions includes, for example, cases in which the wave field is everywhere nonzero and an analytic function of $z = x + iy$.

Demonstrating uniqueness to within a constant is equivalent to showing that the only solutions to the corresponding uniform problems with f or g zero are $\phi =$ constant. In this case the transport-of-intensity equation becomes $L\phi = 0$ and can be written explicitly in the form

$$-\nabla_r \cdot (I\nabla_r\phi) = -I\Delta\phi - \nabla_r\phi \cdot \nabla_r I = 0. \tag{29}$$

For harmonic phase functions this reduces to

$$\nabla_r\phi \cdot \nabla_r I = 0. \tag{30}$$

If I is circularly symmetric and $\nabla_r I \neq 0$, then $\nabla_r I$ will be radial and $\nabla_r\phi$ must be azimuthal. Hence the nontrivial solutions of this equation will be those for which ϕ is dependent only on θ.

In this case, then, we conclude that

$$\phi''(\theta) = 0, \tag{31}$$

and so the only solutions are

$$2\pi\phi(\theta) = \alpha\theta + \beta, \tag{32}$$

where α and β are constants. For continuity of the wave field we require that

$$2\pi\phi(\theta) = 2\pi\phi(\theta + 2\pi) + 2\pi n, \tag{33}$$

where n is an integer, and hence α must be an integer. This is precisely the example presented in Ref. 10.

Thus, if the phase is given by a harmonic function and if the intensity is circularly symmetric, then the only solution of the uniform Neuman problem for the transport-of-intensity equation is that given in Ref. 10. Furthermore, we see that there are many solutions that do not differ from each other by simply 2π (or any other constant), and so these solutions do not simply represent a trivial, nonobservable phase difference.

The above phase ambiguity relates to the nature of the phase rather than to a particular method of its reconstruction. Note that

$$L\phi_m = 0, \qquad \partial_n\phi_m|_\Gamma = 0, \tag{34}$$

where ϕ_m is given in Eq. (28). The important point is that this result does not contradict the uniqueness of the solution to the Neuman problem (24) above, since $I(r = 0) = 0$. Note that by the uniqueness theorems of Subsection 3.A, if we were to modify the intensity distribution so as to remove artificially the zero from the wave field, there would be only one valid solution of the transport-of-intensity equation for the phase. Thus it is the presence of intensity zeros that leads to the plethora of solutions for the phase in the example of Ref. 10.

C. Physical Picture

To examine this phase nonuniqueness from a more physical perspective, consider the two solutions, ϕ_1 and ϕ_2, of Eq. (18). Then we must have

$$\nabla_{\mathbf{r}} \cdot \{I(\mathbf{r}, 0)[\nabla\phi_1(\mathbf{r}, 0) - \nabla\phi_2(\mathbf{r}, 0)]\} = 0. \tag{35}$$

If the solution of Eq. (34) is nontrivial, we can find a vector field $\mathbf{A}$ for which

$$I(\mathbf{r}, 0)\nabla\phi_1(\mathbf{r}, 0) = I(\mathbf{r}, 0)\nabla\phi_2(\mathbf{r}, 0) + \nabla \times \mathbf{A}(\mathbf{r}). \tag{36}$$

Note that $\nabla\times$ appearing in this equation is the three-dimensional curl, whereas $\nabla_{\mathbf{r}}\cdot$ is the two-dimensional divergence. Hence we can find $\mathbf{A}$ of the form

$$\mathbf{A}(\mathbf{r}, z) = A(\mathbf{r})\mathbf{k}. \tag{37}$$

That is, $\mathbf{A}$ points in the z direction. In the example of Ref. 10,

$$A(\mathbf{r}) = m\int_r^\infty \frac{I(t)}{t}\,\mathrm{d}t. \tag{38}$$

Since the difference in the direction vectors is proportional to the curl of another vector, this difference vector

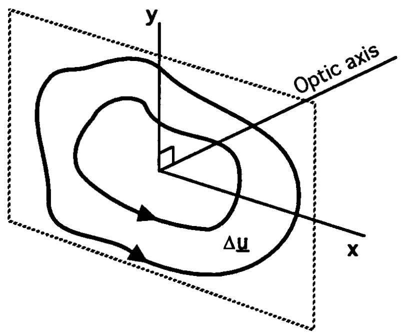

Fig. 3. Identical intensity distributions that can be produced by two fields if the difference in their direction vectors forms closed loops in a plane perpendicular to the optic axis.

forms closed loops in the (x, y) plane. This is analogous to the observation that magnetic-field lines always form closed loops and corresponds to the optical energy flowing in loops in the plane perpendicular to the optic axis (see Fig. 3). In the example of Ref. 10 the difference in the fields can be considered to be produced by energy flowing azimuthally around the optic axis; the phase velocity vector of the wave maps out a helix around the optical axis. Fields displaying this behavior have been discussed in other contexts and are known as optical vortices[17] or screw dislocations.[18] Physical effects of these phenomena have been observed in the form of angular momentum transferred to particles in experiments involving the optical trapping of particles in donut modes.[19] It has also been argued that it would not be possible to construct a smooth flexible mirror able to smooth the phase of a wave field containing a screw dislocation.[20]

One could imagine that there are an infinite number of closed loops possible for the phase velocity vector to trace in the plane perpendicular to the optical axis. In all the cases, however, there must be a phase singularity at the center of the loop, and this is possible only if the intensity is zero at this point.

4. CONCLUSIONS

In this paper we have used the generalized radiance function to obtain a simple derivation and physical picture for the transport-of-intensity equation first given by Teague.[1,2]

We have shown that (1) it is impossible to reconstruct in a unique way incoherent wave fields from intensity measurements alone and (2) although uniqueness theorems exist for the solution to the transport-of-intensity equation for the phase, zeros in the intensity distribution mean that, in general, these theorems are not valid.

We have used the examples of Ref. 10 as a case in which distinct wave fields with distinct phases given by Eq. (28) have identical three-dimensional intensity distributions. These wave fields, therefore, cannot be distinguished by intensity measurements alone. This ambiguity in phase is caused by the presence of the zeros in the intensity distributions, which act as branching points for the phase, and the resulting fields may have very different relative phase distributions.

In other cases in which Ω is simply connected and I is strictly positive, however, the phase function found with the transport-of-intensity equation will be unique and single valued. In applications in adaptive optics it is quite conceivable that these conditions should be met, and efficient algorithms for the solution of Eqs. (23) and (24) under this assumption are currently being explored.

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